Phys 410
Fall 2013
Lecture #3 Summary
10 September, 2013

We considered motion with quadratic drag, $f = -c\, v^2$. The equation of motion is $m\dot{\vec{v}} = m\vec{g} - c\, v^2\hat{v}$. Note that the last term can also be written as $-c\, v\, \vec{v}$, allowing a decomposition into two scalar differential equations: $m\dot{v}_x = 0 - c\sqrt{v_x^2 + v_y^2}\, v_x$ and $m\dot{v}_y = mg - c\sqrt{v_x^2 + v_y^2}\, v_y$. Note that these equations do not separate cleanly into a v_x-only and a v_y-only set of equations, as they did in the linear drag case. The equations are also nonlinear. In this case there is no analytical general solution for this pair of equations. We will consider motion exclusively in the x-direction to simplify the problem.

If we confine the particle to move only in the the x-direction ($v_y = 0$), the first equation reduces to $m\dot{v}_x = -c\, v_x^2$, with solution $v_x = \frac{v_0}{1+t/\tau}$, where we have defined a new characteristic time scale $\tau \equiv m/v_0 c$, and v_0 is the initial velocity. Here we see that the velocity relaxes more slowly than in the linear drag case, where the relaxation was exponential rather than algebraic. The velocity equation can be integrated to find the position of the particle along the x-axis: $x(t) - x(0) = v_0\tau \ln(1 + t/\tau)$. In this case the particle continues to move forever in the x-direction as time increases. Solving the equation for the vertical-only motion is left as a homework problem.

We discussed the motion of a charged particle in a uniform and uni-directional magnetic field $\vec{B}$, subject to the Lorentz force$\vec{F} = q\vec{v} \times \vec{B}$, where q is the charge of the particle. We took $\vec{B} = B\hat{z}$ and found that Newton's second law of motion reduces to three scalar equations: $m\dot{v}_x = qv_yB$, $m\dot{v}_y = -qv_xB$, and $m\dot{v}_z = 0$. The solution for the motion along the magnetic field direction is simple: $z(t) = z_0 + v_{z0}t$, which is uniform motion at constant velocity. We solved the x-y plane motion using the trick of mapping this two-dimensional problem into the complex plane. Define the complex variable $\eta \equiv v_x + iv_y$, where $i = \sqrt{-1}$. The velocity of the particle is now represented as a point in the complex η plane. The pair of coupled differential equations now reduces to a simple equation for the time evolution of η, namely $\dot{\eta} = -i\omega\eta$, and the Cyclotron frequency is defined as $\omega = qB/m$, for the charged particle of mass m.

The equation is solved as $\eta = \eta_0 e^{-i\omega t}$, where $\eta_0 = v_{x0} + iv_{y0} \equiv v_0 e^{i\delta}$. This equation represents uniform circular motion in the η-plane on a circle of radius v_0 starting at an angle δ and rotating clockwise with angular velocity ω. The initial velocities are related to v_0 and δ as $v_{x0} = v_0 cos\delta$ and $v_{y0} = v_0 sin\delta$, and $v_0 = \sqrt{v_{x0}^2 + v_{y0}^2}$, $\delta = \tan^{-1}(v_{y0}/v_{x0})$. The resulting description of the motion can be obtained by taking the real and imaginary parts of η as $v_x(t) = Re[\eta] = v_0\cos(\delta - \omega t)$, and $v_y(t) = Im[\eta] = v_0\sin(\delta - \omega t)$.

The trajectory of the particle in the xy-plane can be solved by a similar method. First define the complex variable $\xi \equiv x + iy$, and relate it to η through the time derivative: $\eta = \dot{\xi}$. Integrate this equation and apply the initial conditions for x and y to obtain $\xi(t) = r_0 e^{i(\phi_0 - \omega t)}$, where the initial positions are written as $x_0 + iy_0 = r_0 e^{i\phi_0}$. The particle motion is described by uniform circular motion around a circle of radius r_0 starting at angle ϕ_0 at angular velocity ω. The resulting motion is three dimensions is helical about the magnetic field axis.

We considered several applications of these ideas to the cyclotron, the Calutron, and Whistlers in the magneto-sphere of the earth.